

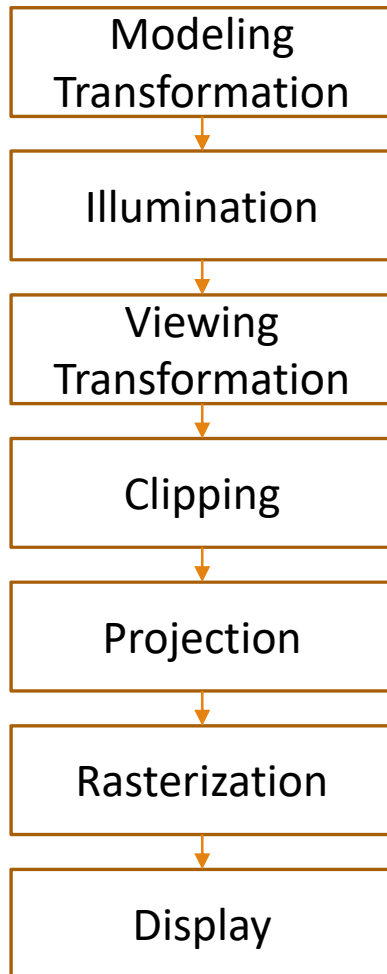
CT4201/EC4215: Computer Graphics

Projections

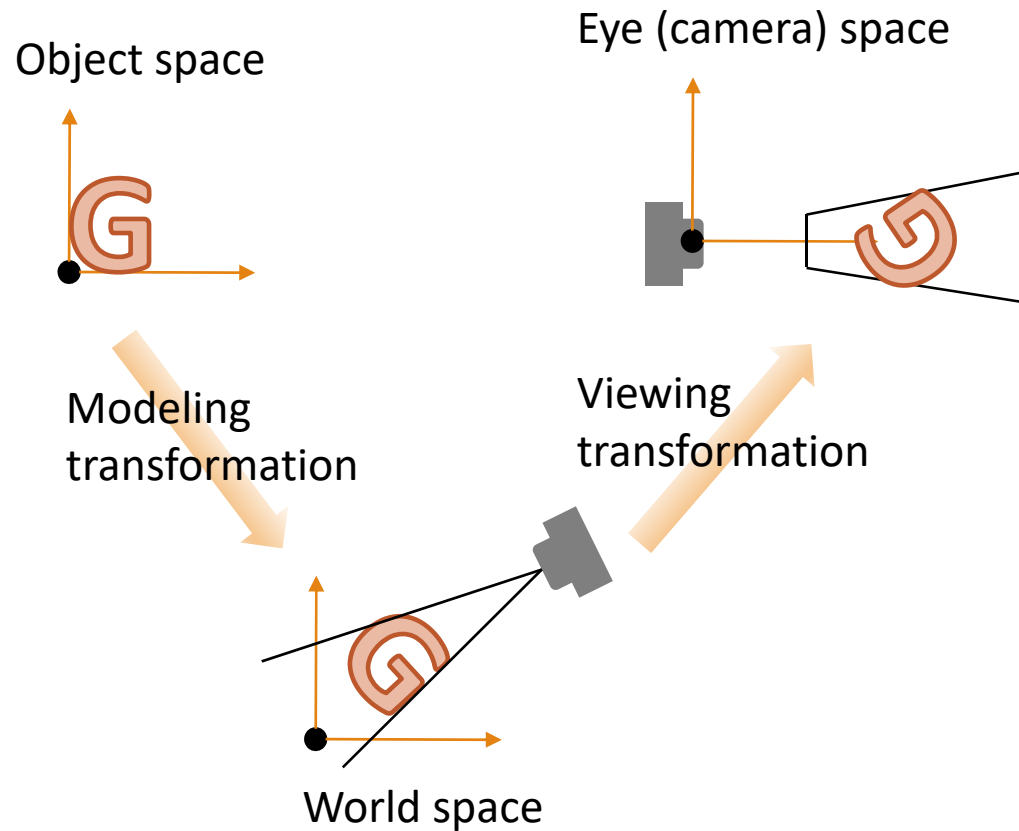
BOCHANG MOON



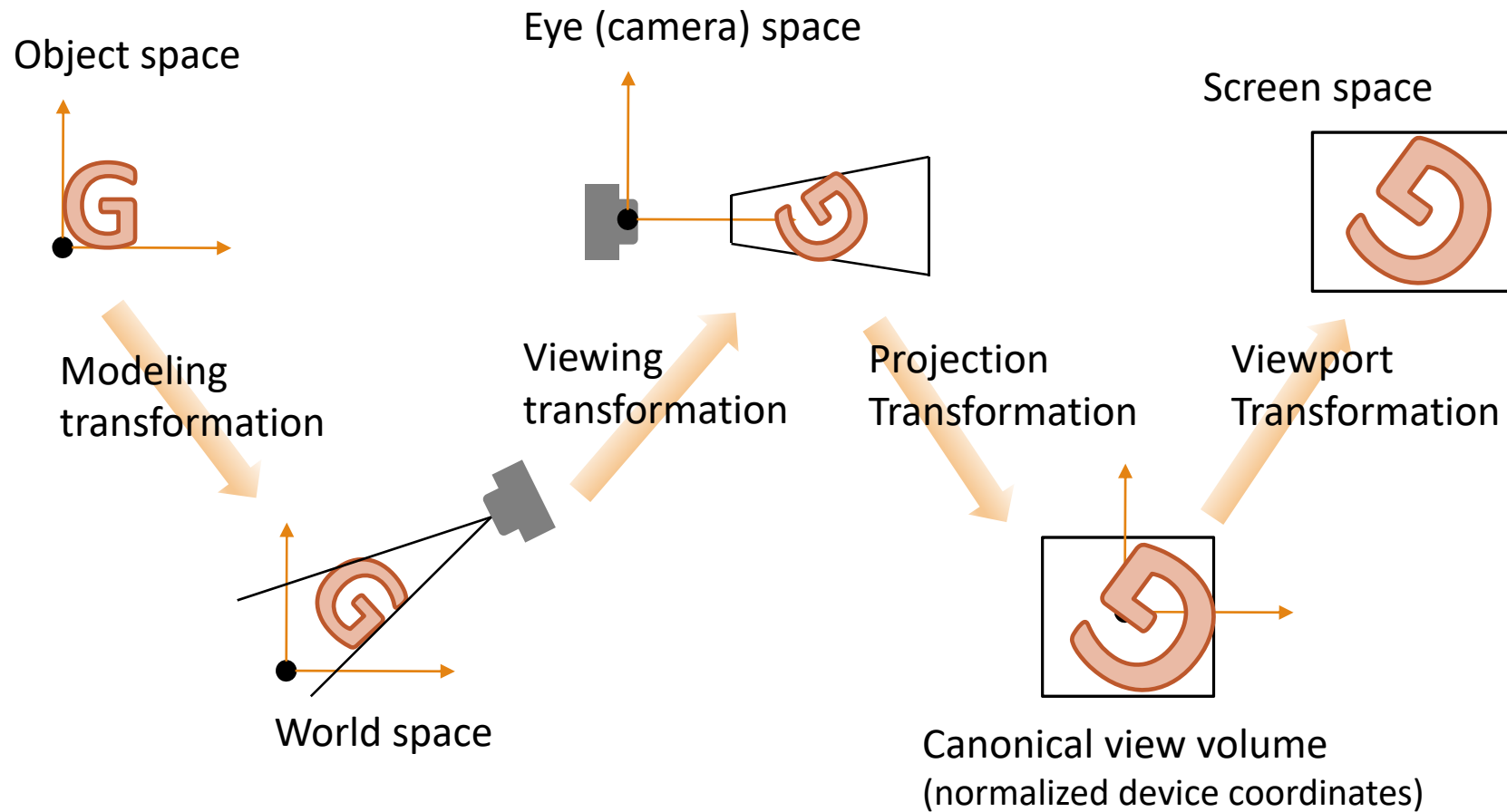
Graphics Pipeline



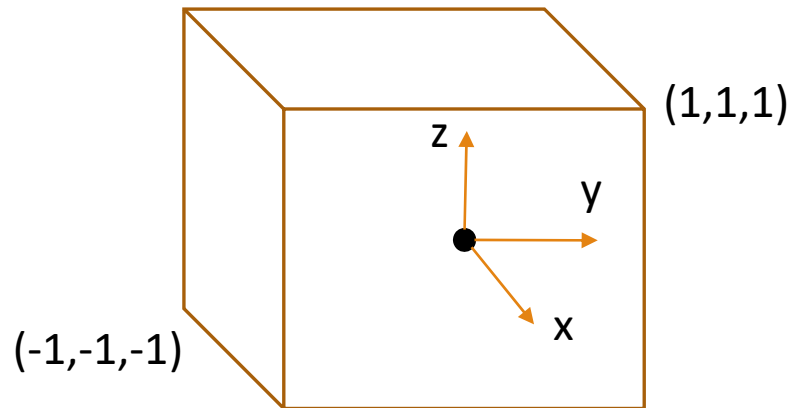
Sequence of Spaces and Transformations



Sequence of Spaces and Transformations

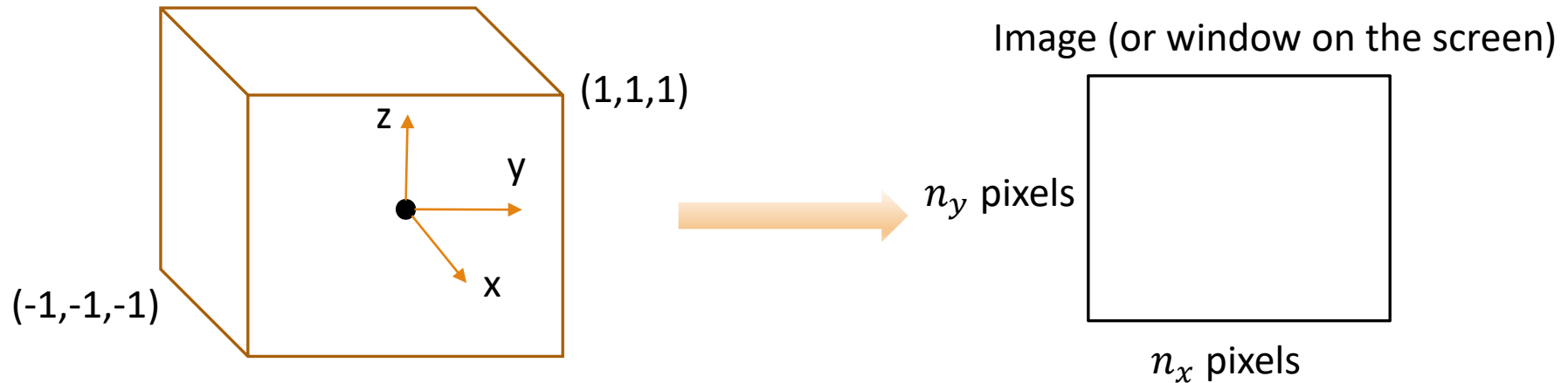


Canonical View Volume



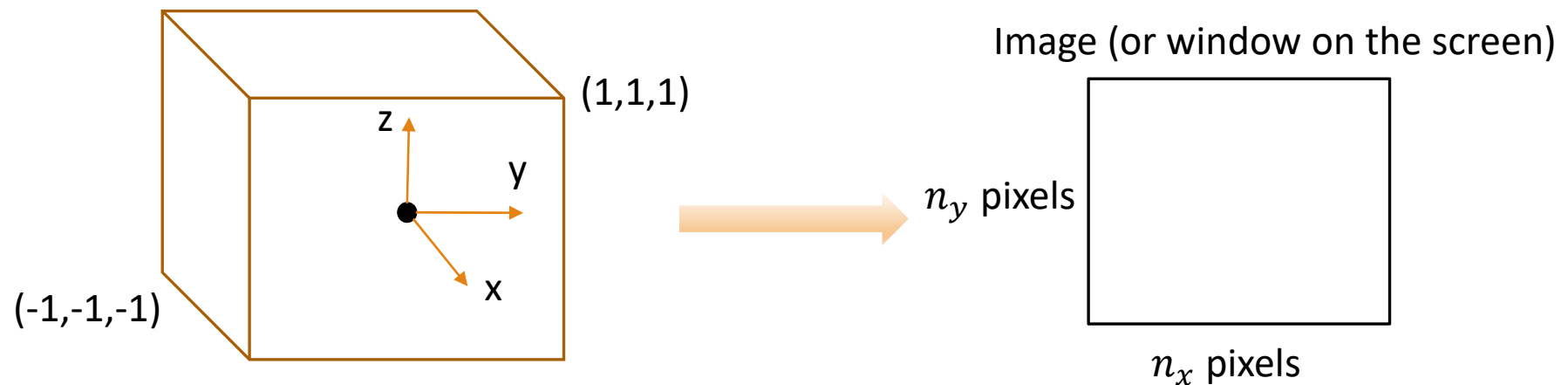
Viewport Transformation

- Primitives (or line segments) within the canonical view volume will be mapped to the image



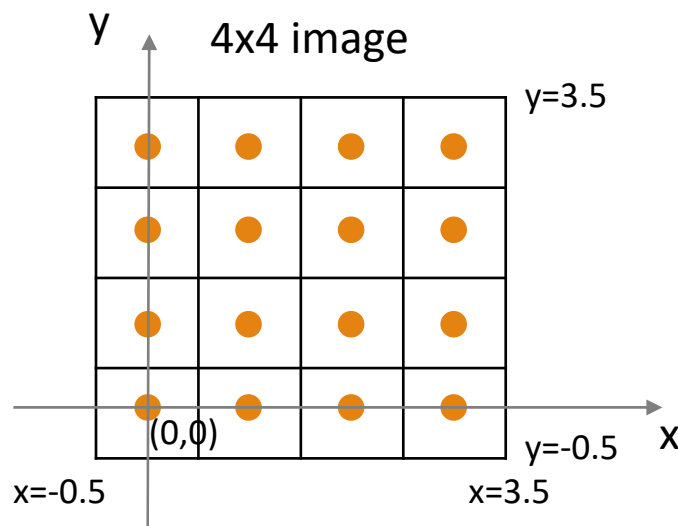
Viewport Transformation

- Ignore the z-coordinates of points for now
 - In practice, we need the z-coordinates and this will be covered later.
- Map the square $[-1,1]^2$ to the rectangle $[-0.5, n_x - 0.5] \times [-0.5, n_y - 0.5]$



Raster Image (again)

- Ignore the z-coordinates of points for now
 - In practice, we need the z-coordinates and this will be covered later.
- Map the square $[-1,1]^2$ to the rectangle $[-0.5, n_x - 0.5] \times [-0.5, n_y - 0.5]$
- Where do we need to locate pixels in 2D space?



Raster Image (again)

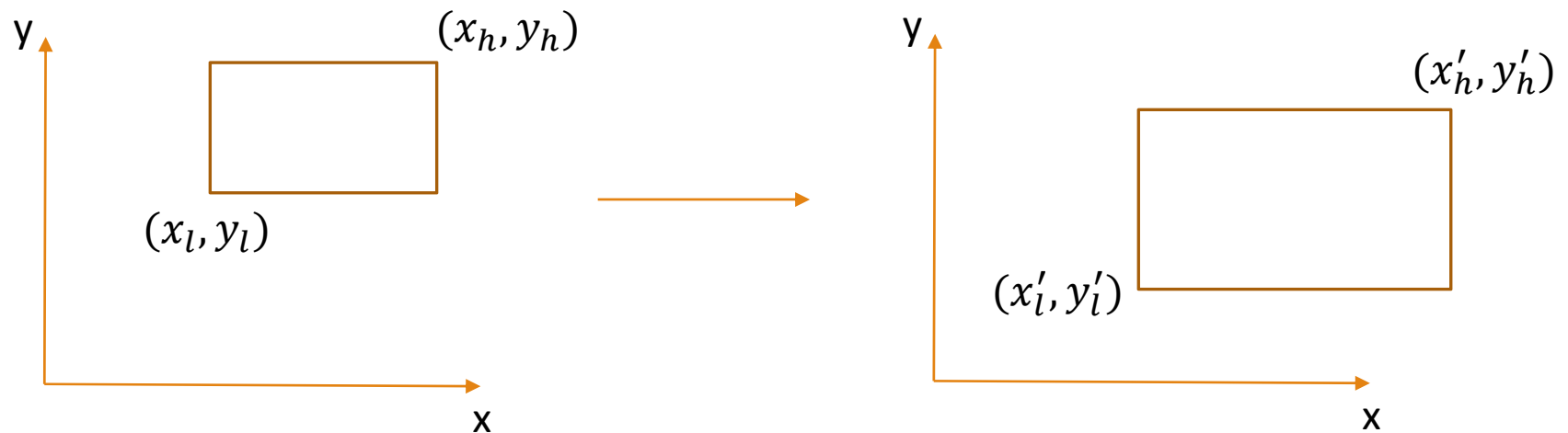
- Ignore the z-coordinates of points for now
 - In practice, we need the z-coordinates and this will be covered later.
- Map the square $[-1,1]^2$ to the rectangle $[-0.5, n_x - 0.5] \times [-0.5, n_y - 0.5]$
- Where do we need to locate pixels in 2D space?
- The rectangular domain of a $n_x \times n_y$ image
 - $R = [-0.5, n_x - 0.5] \times [-0.5, n_y - 0.5]$

Viewport Transformation

- Ignore the z-coordinates of points for now
 - In practice, we need the z-coordinates and this will be covered later.
- Map the square $[-1,1]^2$ to the rectangle $[-0.5, n_x - 0.5] \times [-0.5, n_y - 0.5]$
- Q. How do we transform a rectangle to another rectangle?

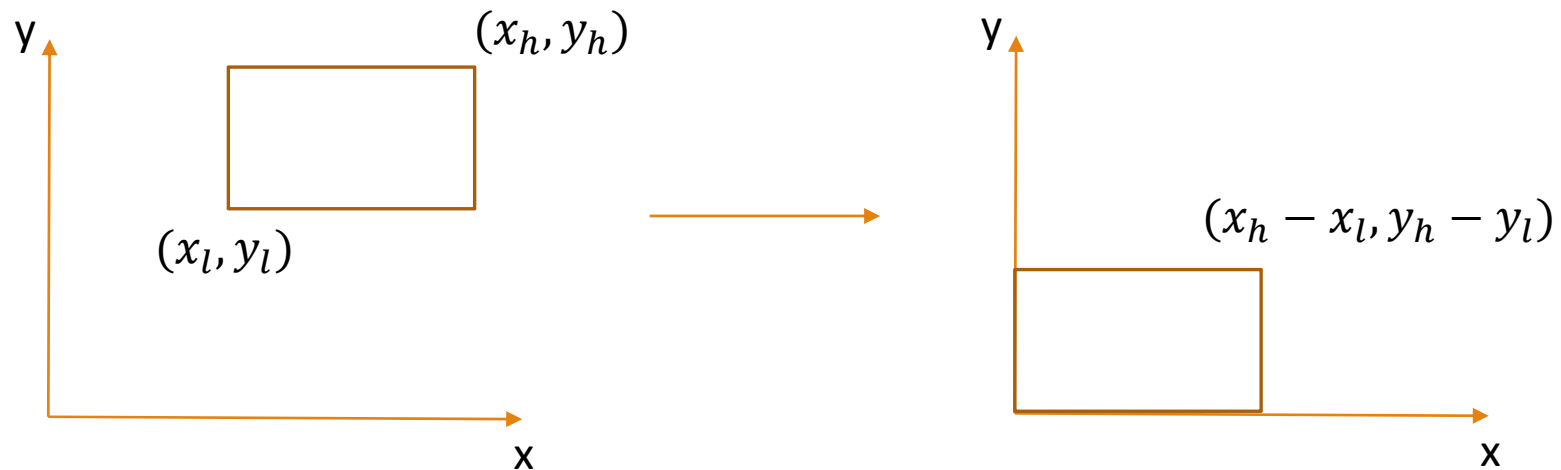
Example: Windowing Transform

- Problem specification: move a 2D rectangle into a new position



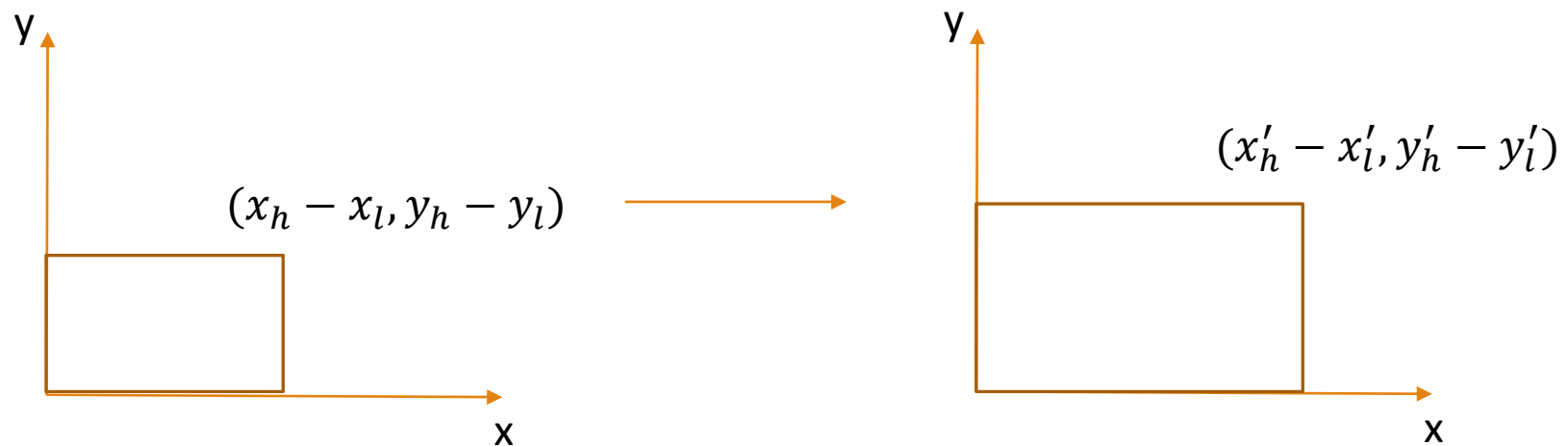
Example: Windowing Transform

- Problem specification: move a 2D rectangle into a new position
 - Step1. translate: move the point (x_l, y_l) to the origin



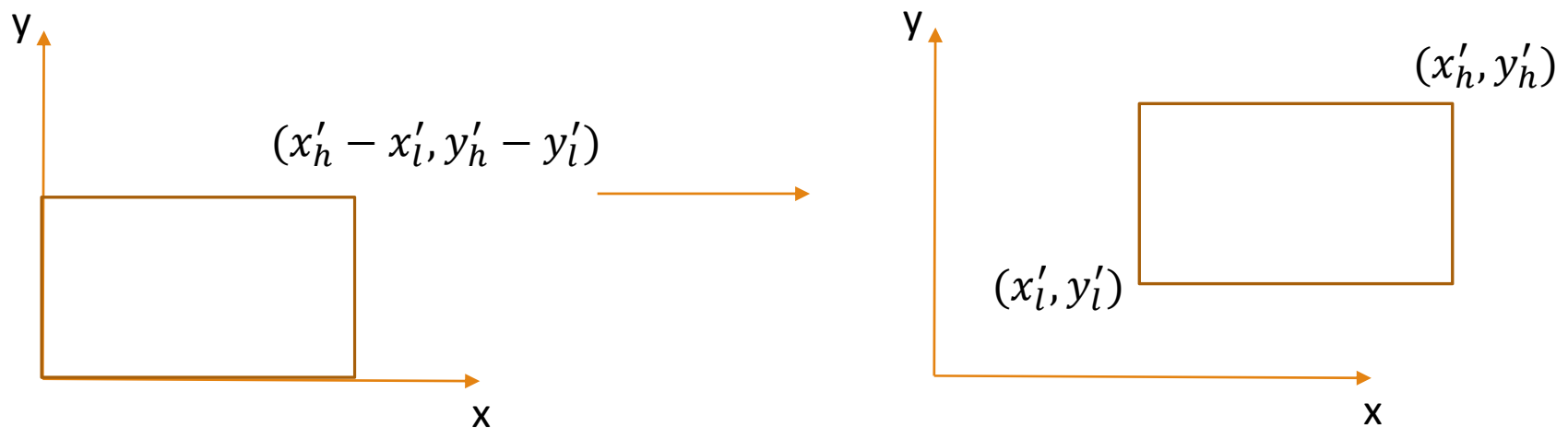
Example: Windowing Transform

- Problem specification: move a 2D rectangle into a new position
 - Step2. scale: resize the rectangle to be the same size of the target.



Example: Windowing Transform

- Problem specification: move a 2D rectangle into a new position
 - Step3. translate: move the origin to point (x'_i, y'_i)



Example: Windowing Transform

- Problem specification: move a 2D rectangle into a new position
 - Target = $\text{translate}(x'_l, y'_l) \text{ scale} \left(\frac{x'_h - x'_l}{x_h - x_l}, \frac{y'_h - y'_l}{y_h - y_l} \right) \text{ translate}(-x_l, -y_l)$

$$\circ = \begin{bmatrix} 1 & 0 & x'_l \\ 0 & 1 & y'_l \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{x'_h - x'_l}{x_h - x_l} & 0 & 0 \\ 0 & \frac{y'_h - y'_l}{y_h - y_l} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_l \\ 0 & 1 & -y_l \\ 0 & 0 & 1 \end{bmatrix}$$

$$\circ = \begin{bmatrix} \frac{x'_h - x'_l}{x_h - x_l} & 0 & \frac{x'_l x_h - x'_h x_l}{x_h - x_l} \\ 0 & \frac{y'_h - y'_l}{y_h - y_l} & \frac{y'_l y_h - y'_h y_l}{y_h - y_l} \\ 0 & 0 & 1 \end{bmatrix}$$

Viewport Transformation

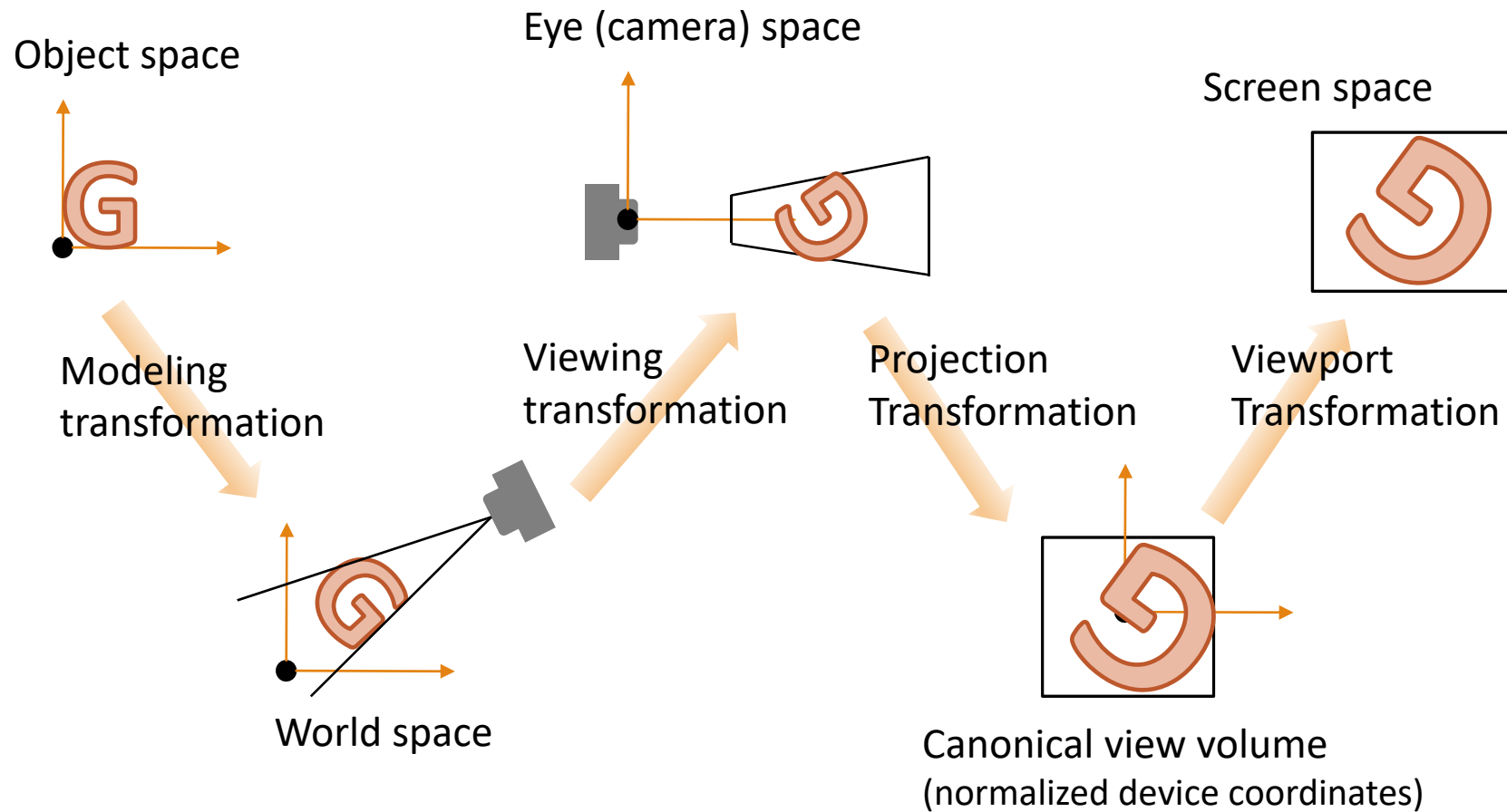
- Ignore the z-coordinates of points for now
 - In practice, we need the z-coordinates and this will be covered later.
- Map the square $[-1,1]^2$ to the rectangle $[-0.5, n_x - 0.5] \times [-0.5, n_y - 0.5]$

- $$\begin{bmatrix} x_{screen} \\ y_{screen} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{n_x}{2} & 0 & \frac{n_x-1}{2} \\ 0 & \frac{n_y}{2} & \frac{n_y-1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{canonical} \\ y_{canonical} \\ 1 \end{bmatrix}$$

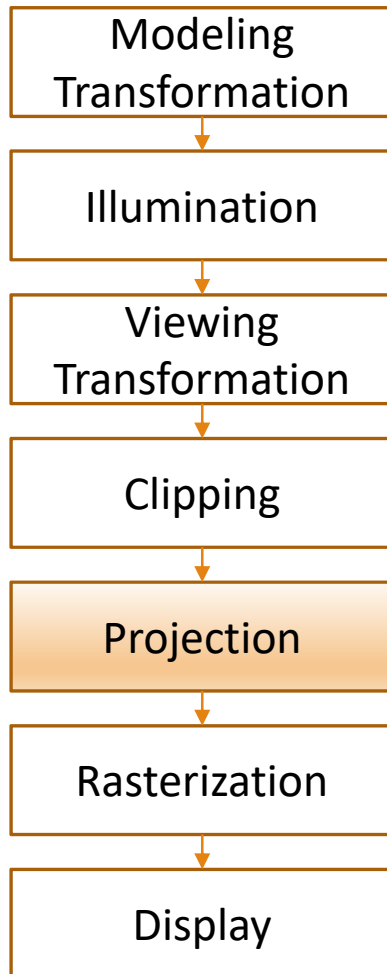
- For the case with z-coordinates,

- $$M_{viewport} = \begin{bmatrix} \frac{n_x}{2} & 0 & 0 & \frac{n_x-1}{2} \\ 0 & \frac{n_y}{2} & 0 & \frac{n_y-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

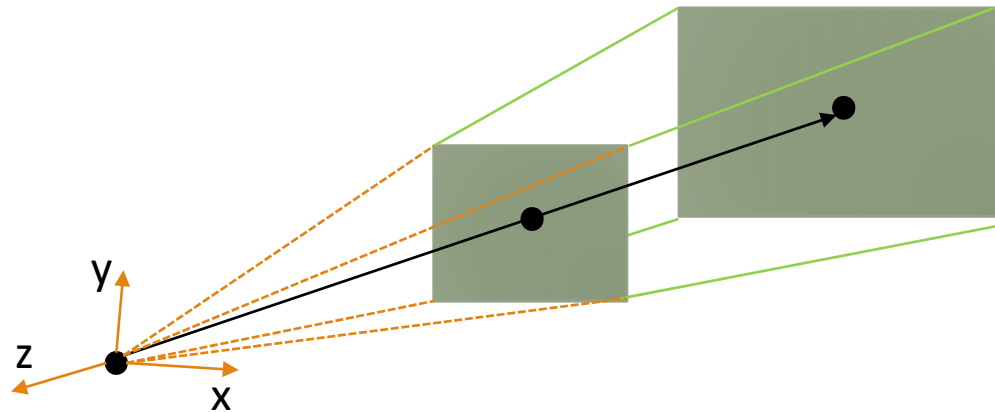
Sequence of Spaces and Transformations



Projections



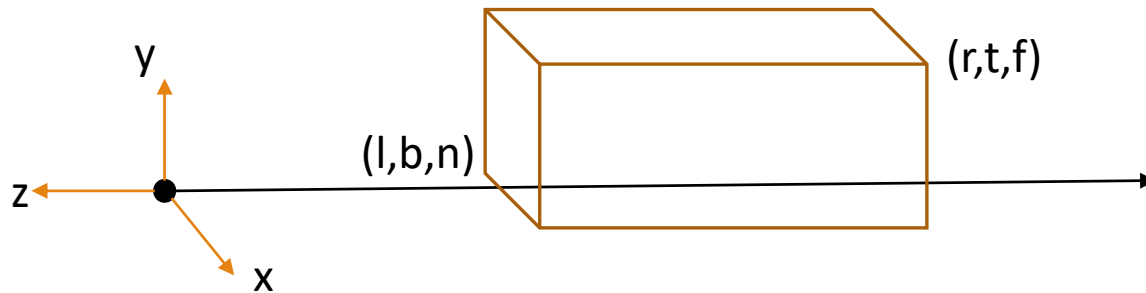
- Transform 3D points in *eye space* to 2D points in *image space*



- Two types of projections
 - Orthographic projection
 - Perspective projection

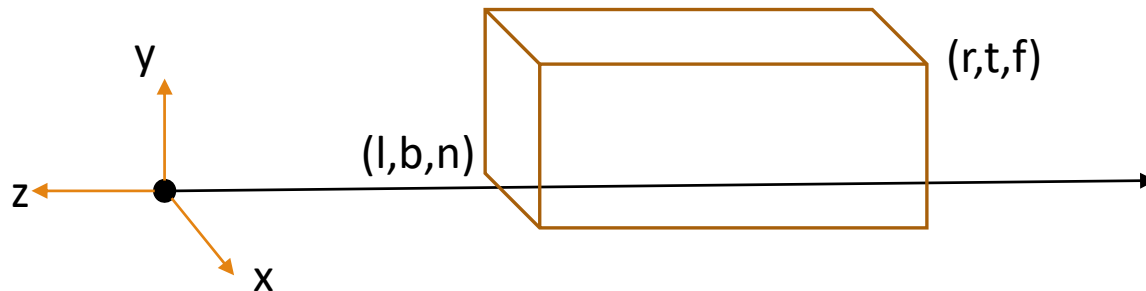
Orthographic Projection

- Assumption
 - A viewer is looking along the minus z-axis with his head pointing in the y-direction
 - Implies $n > f$



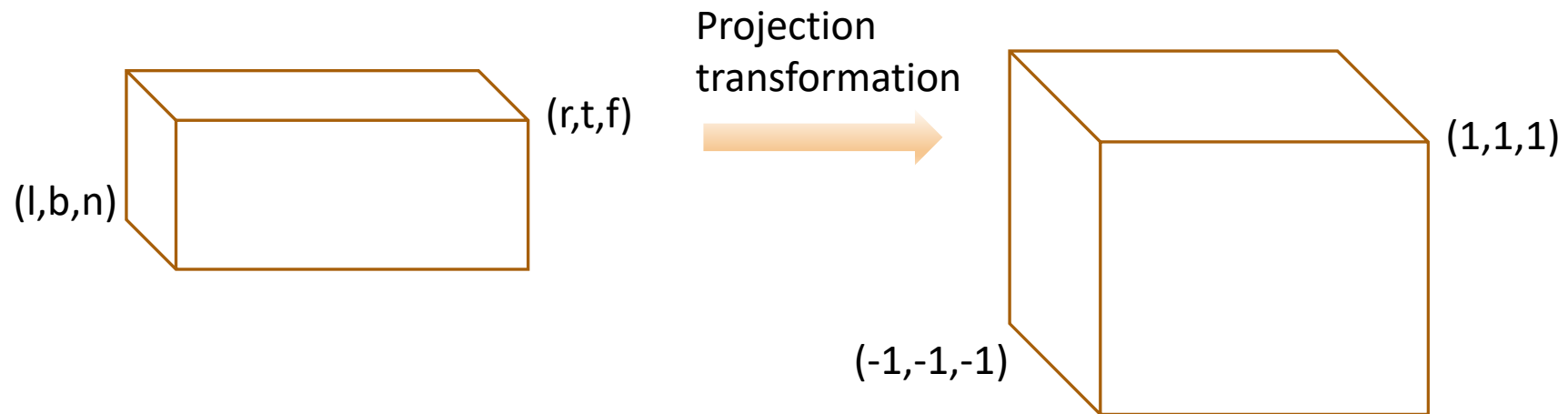
Orthographic Projection

- The view volume (*orthographic view volume*) is an axis-aligned box
 - $[l, r] \times [b, t] \times [f, n]$
- Notations
 - $x = l \equiv$ left plane, $x = r \equiv$ right plane
 - $y = b \equiv$ bottom plane, $y = t \equiv$ top plane
 - $z = n \equiv$ near plane, $z = f \equiv$ far plane



Orthographic Projection

- Transform points in orthographic view volume to the canonical view volume
 - Also windowing transform (3D)



Orthographic Projection

- Transform points in orthographic view volume to the canonical view volume
 - Also windowing transform (3D)
 - Map a box $[x_l, x_h] \times [y_l, y_h] \times [z_l, z_h]$ to another box $[x'_l, x'_h] \times [y'_l, y'_h] \times [z'_l, z'_h]$

$$\begin{bmatrix} \frac{x'_h - x'_l}{x_h - x_l} & 0 & 0 & \frac{x'_l x_h - x'_h x_l}{x_h - x_l} \\ 0 & \frac{y'_h - y'_l}{y_h - y_l} & 0 & \frac{y'_l y_h - y'_h y_l}{y_h - y_l} \\ 0 & 0 & \frac{z'_h - z'_l}{z_h - z_l} & \frac{z'_l z_h - z'_h z_l}{z_h - z_l} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orthographic Projection

- Transform points in orthographic view volume to the canonical view volume
 - Also windowing transform (3D)

- $M_{ortho} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Composite Transformation

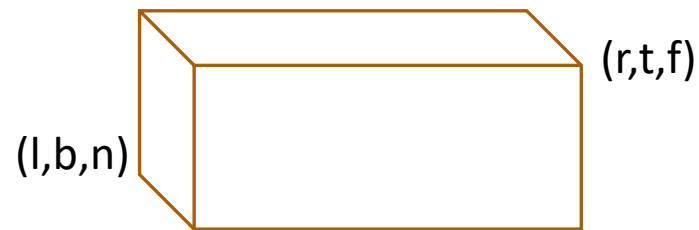
- The matrix that transforms points in world space to screen coordinate:
- $M = M_{viewport}M_{ortho}M_{viewing}$

Orthographic Projection

- Transform points in orthographic view volume to the canonical view volume
 - Also windowing transform (3D)
- Tend to ignore relative distances between objects and eye
 - Unrealistic
- In practice,
 - We usually do not use this projection.
 - It can be useful in applications where relative lengths should be judged.

Orthographic Projection in OpenGL

- `void glOrtho(GLdouble left, GLdouble right,`
- `GLdouble bottom, GLdouble top,`
- `GLdouble nearVal, GLdouble farVal);`



Perspective Projection

- Objects in an image become smaller as their distance from the eye increases.
- History of perspective:
 - Artists from the Renaissance period employed the perspective property.



Images from wikipedia

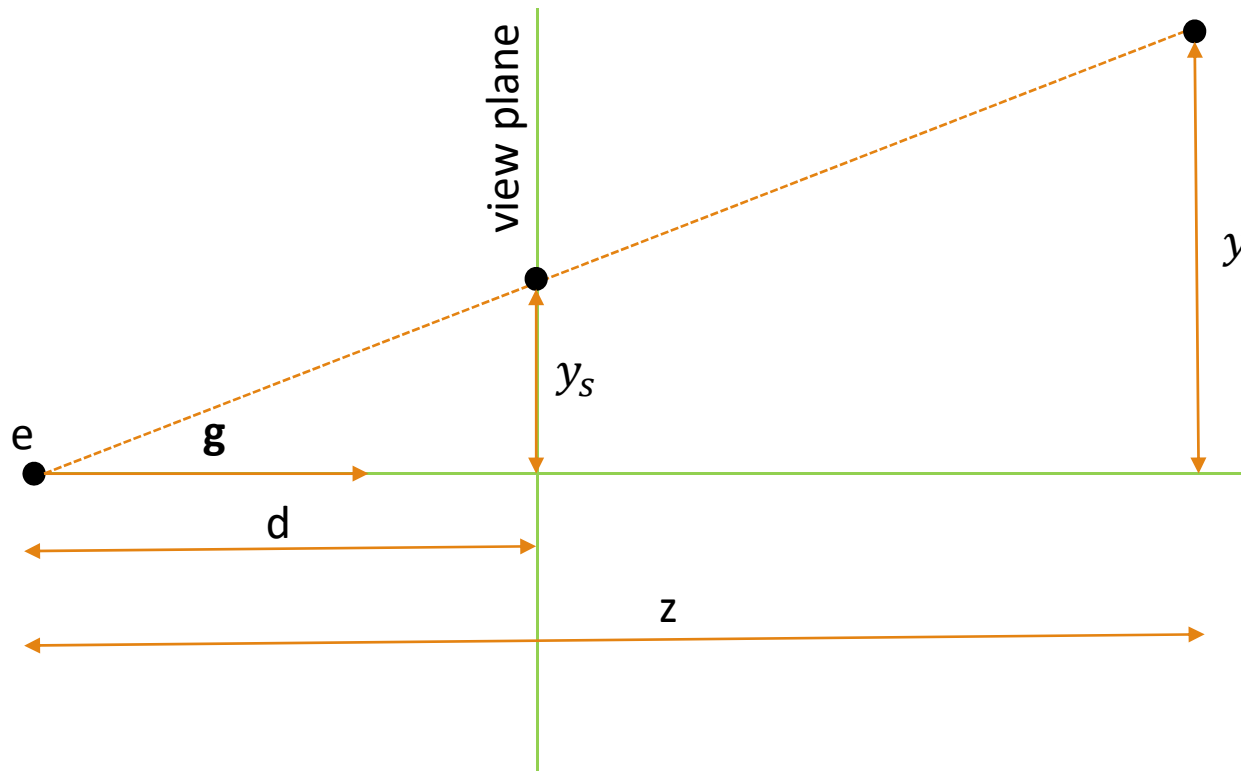
Perspective Projection

- Objects in an image become smaller as their distance from the eye increases.
- History of perspective:
 - Artists from the Renaissance period employed the perspective property.
- In everyday life?



Perspective Projection

- $y_s = \frac{d}{z} y$
 - y_s : y-axis coordinate in view plane
 - y : distance of the point along the y-axis



Homogeneous Coordinate

- Represent a point (x, y, z) with an extra coordinate w
 - (x, y, z, w)
 - In the previous lecture, $w = 1$
- Let's define w to be the denominator of the x -, y -, z -coordinates
 - (x, y, z, w) represent the 3D point $(\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$
 - A special case, $w = 1$, is still valid.
 - w can be any values

Projective Transform

- Let's define w to be the denominator of the x -, y -, z -coordinates
 - (x, y, z, w) represent the 3D point $(\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$
 - A special case, $w = 1$, is still valid.
 - w can be any values

- Projective transformation

- $$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ e & f & g & h \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- $(x', y', z') = (\frac{\tilde{x}}{\tilde{w}}, \frac{\tilde{y}}{\tilde{w}}, \frac{\tilde{z}}{\tilde{w}})$

Perspective Projection

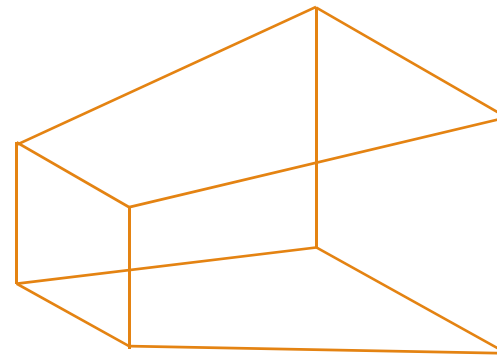
- Example with 2D homogeneous vector $[y \ z \ 1]^T$
 - $\begin{bmatrix} y_s \\ 1 \end{bmatrix} = \begin{bmatrix} d & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix}$
 - This is corresponding to the perspective equation, $y_s = \frac{d}{z}y$.

Perspective Projection

- Some info. for perspective matrix
 - Define our project plane as the near plane
 - Distance to the near plane: $-n$
 - Distance to the far plane: $-f$
- Perspective equation: $y_s = \frac{n}{z} y$

- *Perspective matrix*

- $$P = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n + f & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



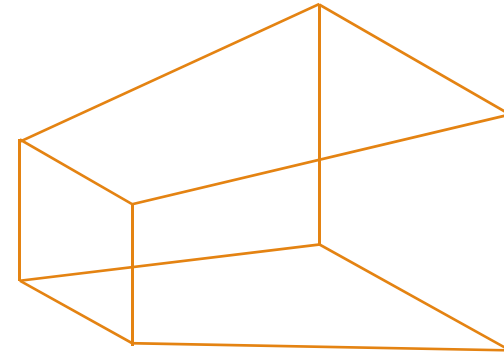
Perspective Projection

- *Perspective matrix*

- $$P = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n + f & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- A mapping with the perspective matrix:

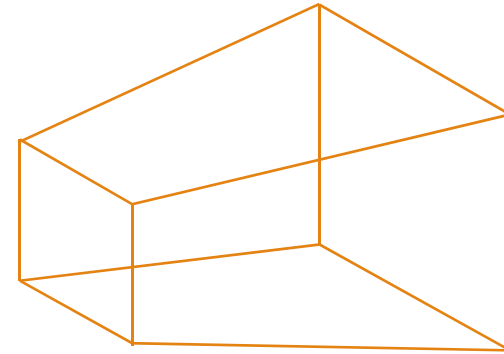
- $$P \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} nx \\ ny \\ (n + f)z - fn \\ z \end{bmatrix} = \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n + f - \frac{fn}{z} \\ 1 \end{bmatrix}$$



Perspective Projection

- A mapping with the perspective matrix:

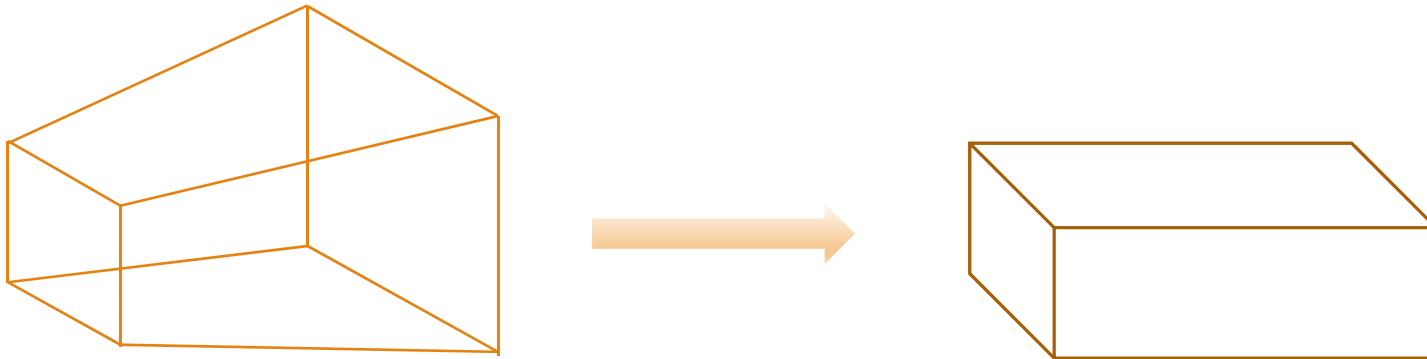
- $$P \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} nx \\ ny \\ (n+f)z - fn \\ z \end{bmatrix} = \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n + f - \frac{fn}{z} \\ 1 \end{bmatrix}$$



- Properties
 - The first, second, and fourth rows are for the perspective equation.
 - The third row is for keeping z coordinate at least approximately.
 - E.g., when $z = n$, transformed z coordinate is still n .
 - E.g., when $z > n$, we cannot preserve the z coordinate exactly, but relative orders between points will be preserved.

Perspective Projection

- Perspective matrix
 - Map the perspective view volume to the orthographic view volume.



Composite Transformation

- The matrix that transforms points in world space to screen coordinate:

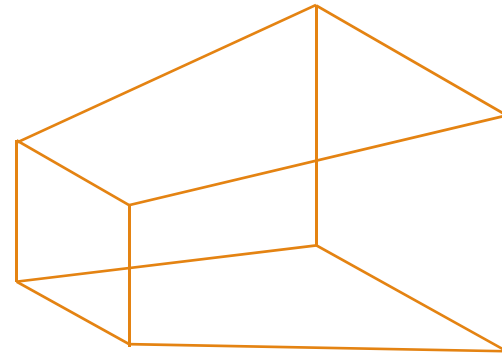
- $M = M_{viewport}M_{ortho}PM_{viewing} = M_{viewport}M_{per}M_{viewing}$

- $M_{per} = M_{ortho}P$ (*perspective projection matrix*)

- $M_{per} = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{l+r}{l-r} & 0 \\ 0 & \frac{2n}{t-b} & \frac{b+t}{b-t} & 0 \\ 0 & 0 & \frac{f+n}{n-f} & \frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Perspective Projection in OpenGL

- `void glFrustum(GLdouble left, GLdouble right,`
- `GLdouble bottom, GLdouble top,`
- `GLdouble nearVal, GLdouble farVal);`



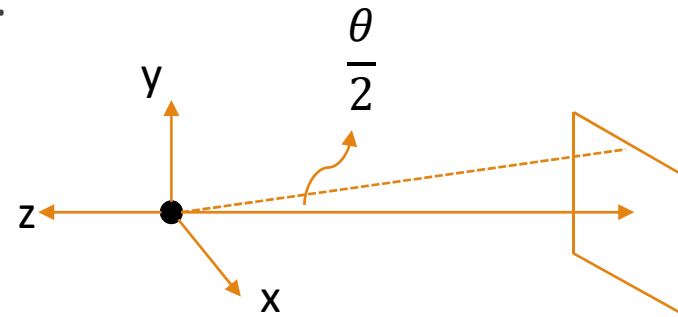
Perspective Projection in OpenGL

- `void gluPerspective(GLdouble fovy, GLdouble aspect,`
- `GLdouble zNear, GLdouble zFar);`

- Parameters
 - `fovy`: field of view (in degrees) in the y direction
 - `aspect`: aspect ratio is the ratio of x (width) to y (height)

- Symmetric constraints are implicitly applied.
 - $l = -r, b = -t$

- A constraint to prevent image distortion
 - $\frac{n_x}{n_y} = \frac{r}{t}$



Further Reading

- In our textbook, Fundamentals of Computer Graphics (4th edition)
 - Chapter 7