CT5202: Photorealistic Rendering

Monte Carlo Integration

Lecturer: Bochang Moon

Background: Probability

- $Probability(x \in [a,b]) = \int_a^b p(x)\,dx$
  - $p(x)$ is the probability density function (pdf)
  - $p(x) \geq 0$
  - $\int_{-\infty}^{+\infty} p(x)\,dx = 1$

- Example: a random variable ξ with uniform probability
  - $q(\xi) = \begin{matrix} 1 & if\ 0 \leq \xi < 1 \\ 0 & otherwise \end{matrix}$
  - $Probability(a \leq \xi \leq b) = \int_a^b 1\,dx = b - a,$
  - where $[a,b] \in [0,1)$
  -

Background: Expected Value

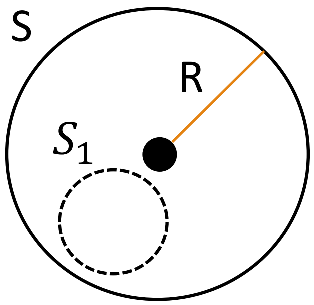
- Expected value $E(f(x))$
  - Average value of a function $f(x)$

- $E(f(x)) = \int f(x)p(x)dx$
  - x is the random variable with a pdf p(x)

- Linearity:
  - $E(x + y) = E(x) + E(y)$
  - $E(f(x) + g(y)) = E(f(x)) + E(g(y))$
    - A function of a random variable is also a random variable
    - This holds for the random variables, which are both independent and dependent

Background: Multidimensional Random Variables

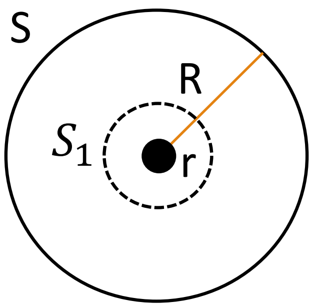
- Pdf $p: S \rightarrow \mathbb{R}$
  - S: a space has a measure μ
  - e.g., S: surface of a sphere, μ: area
- x is a random variable with $x \sim p$
- Probability that x will take from a region $S_i \in S$
  - $Probability(x \in S_i) = \int_{S_i} p(x) d\mu$
- e.g., a uniformly distributed random variable $\alpha \in S$
  - $p(\alpha) = \frac{1}{\pi R^2}$
- e.g., probability that α is in a region $S_1 \subset S$
  - $Probability(\alpha \in S_1) = \int_{S_1} \frac{1}{\pi R^2} dA$



Background: Multidimensional Random Variables

- e.g., given a uniformly distributed random variable $\alpha \in S$,

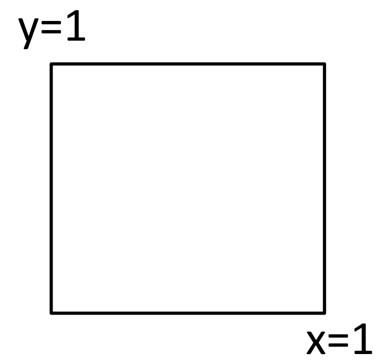
what's probability that α is in a region $S_1 \subset S$?



  - $\alpha = (r, \phi)$ in polar coordinates
  - $r = \frac{R}{2}$
  - Differential area $dA = r\ dr\ d\phi$
  - $Probability(\alpha \in S_1) = \int_0^{2\pi} \int_0^{\frac{R}{2}} \frac{1}{\pi R^2} r\ dr\ d\phi = 0.25$

Background: Multidimensional Random Variables

- e.g., what's the expected value of the x coordinate:
  - $p(x, y) = 4xy$
  - $S = [0,1] \times [0,1]$
  - $E(x) = \int_S f(x, y)p(x, y)dA$
  - $= \int_0^1 \int_0^1 4x^2 y dx dy \qquad (\because f(x, y) = x)$
  - $= \frac{2}{3}$



Background: Variance

- Variance of a random variable x
  - $V(x) \equiv E([x - E(x)]^2) = E(x^2) - [E(x)]^2$

- Standard deviation
  - $\sigma(x) = \sqrt{V(x)}$

- Properties:
  - $V(ax) = a^2 V(x)$
  - $V(x + a) = V(x)$
  - $V(ax + by) = a^2 V(x) + b^2 V(y) + 2abCov(X, Y)$
    - If x and y are independent, $Cov(X, Y) = 0$

Background: Estimated Mean

- Suppose independent and identically distributed (iid) random variables
  - Random variables x_i are independent and have a common pdf p

  - Estimated mean = an estimate of the expectation $E(x)$
    - $E(x) \approx \frac{1}{N}\sum_{i=1}^{N} x_i$

    - As N increases, the variance of this estimate decreases. Why?
      - $V(x_i) = \sigma^2$
      - $V\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right) = \frac{1}{N^2}\sum_{i=1}^{N} V(x_i) = \frac{1}{N}\sigma^2$

Background: Estimated Mean

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  - Estimated mean = an estimate of the expectation $E(x)$
    - $E(x) \approx \frac{1}{N}\sum_{i=1}^{N} x_i$
    - As N increases, the variance of this estimate decreases. Why?
  - Q. What's the estimation error?
  - Law of Large Numbers
    - $Probability\left[E(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i\right] = 1$

Monte Carlo Integration

- Given a function $f: S \rightarrow \mathbb{R}$, a random variable $x \sim p$
  - Expected value of $f(x)$
    - $E\big(f(x)\big) = \int_{x \in S} f(x)p(x)d\mu \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i)$
  - By substituting $g = fp$
    - $E\big(f(x)\big) = \int_{x \in S} g(x)d\mu \approx \frac{1}{N}\sum_{i=1}^{N} \frac{g(x_i)}{p(x_i)}$
  - How can we apply this technique to the transport equation?
    - $L_s(k_o) = \int_{all\ x'} \frac{\rho(k_i,k_o)L_s(x',x-x')v(x,x')cos\theta_i cos\theta_{'}}{||x-x'||^2} dA'$