

CT5202: Photorealistic Rendering

Monte Carlo Integration

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Background: Probability

- $Probability(x \in [a, b]) = \int_a^b p(x) dx$
 - $p(x)$ is the probability density function (pdf)
 - $p(x) \geq 0$
 - $\int_{-\infty}^{+\infty} p(x) dx = 1$
- Example: a random variable ξ with uniform probability
 - $q(\xi) = \begin{cases} 1 & \text{if } 0 \leq \xi < 1 \\ 0 & \text{otherwise} \end{cases}$
 - $Probability(a \leq \xi \leq b) = \int_a^b 1 dx = b - a,$
 - where $[a, b] \in [0, 1)$
 -

Background: Expected Value

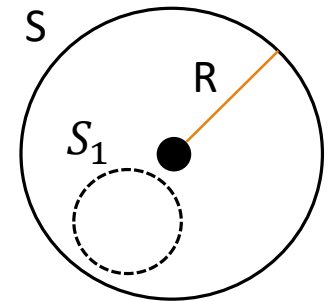
- Expected value $E(f(x))$
 - Average value of a function $f(x)$

- $E(f(x)) = \int f(x)p(x)dx$
 - x is the random variable with a pdf $p(x)$

- Linearity:
 - $E(x + y) = E(x) + E(y)$
 - $E(f(x) + g(y)) = E(f(x)) + E(g(y))$
 - A function of a random variable is also a random variable
 - This holds for the random variables, which are both independent and dependent

Background: Multidimensional Random Variables

- Pdf $p: S \rightarrow \mathbb{R}$
 - S : a space has a measure μ
 - e.g., S : surface of a sphere, μ : area
- x is a random variable with $x \sim p$
- Probability that x will take from a region $S_i \in S$
 - $Probability(x \in S_i) = \int_{S_i} p(x) d\mu$
- e.g., a uniformly distributed random variable $\alpha \in S$
 - $p(\alpha) = \frac{1}{\pi R^2}$
- e.g., probability that α is in a region $S_1 \subset S$
 - $Probability(\alpha \in S_1) = \int_{S_1} \frac{1}{\pi R^2} dA$

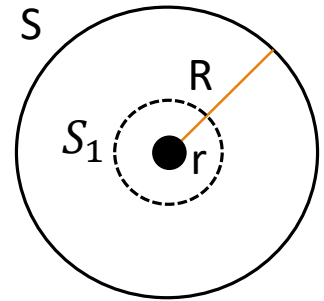


Background: Multidimensional Random Variables

- e.g., given a uniformly distributed random variable $\alpha \in S$,

what's probability that α is in a region $S_1 \subset S$?

- $\alpha = (r, \phi)$ in polar coordinates
- $r = \frac{R}{2}$
- Differential area $dA = r dr d\phi$
- $Probability(\alpha \in S_1) = \int_0^{2\pi} \int_0^{\frac{R}{2}} \frac{1}{\pi R^2} r dr d\phi = 0.25$



Background: Multidimensional Random Variables

- e.g., what's the expected value of the x coordinate:

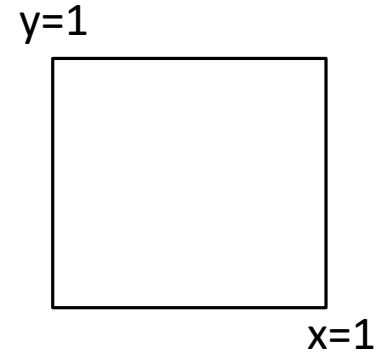
- $p(x, y) = 4xy$

- $S = [0,1] \times [0,1]$

- $E(x) = \int_S f(x, y)p(x, y)dA$

- $= \int_0^1 \int_0^1 4x^2y dx dy \quad (\because f(x, y) = x)$

- $= \frac{2}{3}$



Background: Variance

- Variance of a random variable x
 - $V(x) \equiv E([x - E(x)]^2) = E(x^2) - [E(x)]^2$
- Standard deviation
 - $\sigma(x) = \sqrt{V(x)}$
- Properties:
 - $V(ax) = a^2V(x)$
 - $V(x + a) = V(x)$
 - $V(ax + by) = a^2V(x) + b^2V(y) + 2abCov(X, Y)$
 - If x and y are independent, $Cov(X, Y) = 0$

Background: Estimated Mean

- Suppose independent and identically distributed (iid) random variables
 - Random variables x_i are independent and have a common pdf p
 - Estimated mean = an estimate of the expectation $E(x)$
 - $E(x) \approx \frac{1}{N} \sum_{i=1}^N x_i$
 - As N increases, the variance of this estimate decreases. Why?
 - $V(x_i) = \sigma^2$
 - $V\left(\frac{1}{N} \sum_{i=1}^N x_i\right) = \frac{1}{N^2} \sum_{i=1}^N V(x_i) = \frac{1}{N} \sigma^2$

Background: Estimated Mean

- Suppose independent and identically distributed (iid) random variables
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 - $E(x) \approx \frac{1}{N} \sum_{i=1}^N x_i$
 - As N increases, the variance of this estimate decreases. Why?
 - Q. What's the estimation error?
 - Law of Large Numbers
 - *Probability* $\left[E(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N x_i \right] = 1$

Monte Carlo Integration

- Given a function $f: S \rightarrow \mathbb{R}$, a random variable $x \sim p$
 - Expected value of $f(x)$
 - $E(f(x)) = \int_{x \in S} f(x)p(x)d\mu \approx \frac{1}{N} \sum_{i=1}^N f(x_i)$
 - By substituting $g = fp$
 - $E(f(x)) = \int_{x \in S} g(x)d\mu \approx \frac{1}{N} \sum_{i=1}^N \frac{g(x_i)}{p(x_i)}$
 - How can we apply this technique to the transport equation?
 - $L_S(k_o) = \int_{all\ x'} \frac{\rho(k_i, k_o) L_S(x', x-x') v(x, x') \cos\theta_i \cos\theta'}{\|x-x'\|^2} dA'$