CT5202: Photorealistic Rendering

Density Estimation for Rendering Techniques

BOCHANG MOON

Density Estimation

- $P(a < X < b) = \int_a^b f(x)dx$ for all a < b

- Problem:
  - Input: a set of observed data points generated from an unknown probability density function (pdf)
  - Output: an estimate of the pdf

- A parametric approach:
  - We assume the pdf is a known parametric family of distributions
  - e.g., if we assume the samples are drawn from a normal distribution, we can estimate the mean μ and variance σ^2 with a sample mean and sample variance

Density Estimation

- In practice,
  - It is hard to employ the parametric approach as our target function (e.g., radiance) may be very complex

- Nonparametric approaches:
  - Histograms
  - Kernel estimator
  - Nearest neighbor method

Histograms

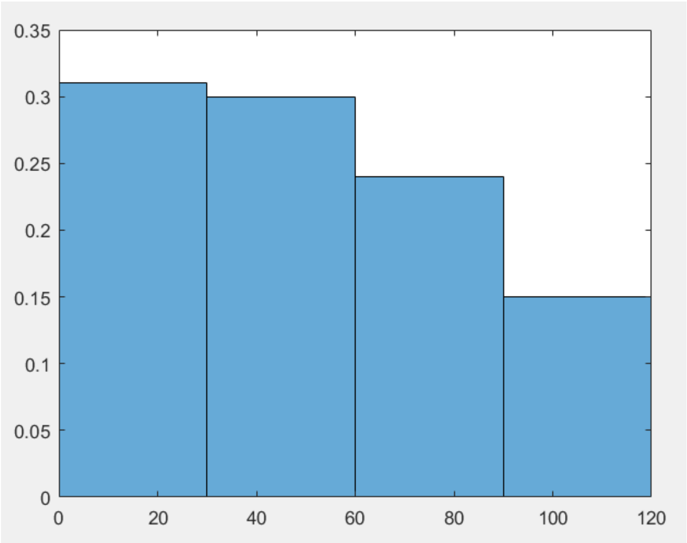
- Input: *n* real data $X_1, \dots, X_n$
- Given an origin x_0 and a bin width *h*, the bins of histograms are defined by the intervals $[x_0 + mh, x_0 + (m+1)h)$, where m is an integer
- $\hat{f}(x) = \frac{\#\, of\ X_i\ in\ same\ bin\ as\ x}{nh}$

Data

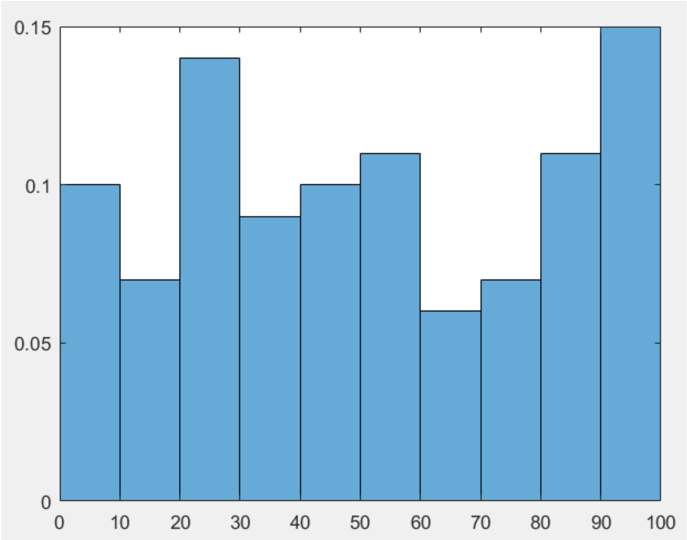
- Example: 100 random samples from a unknown pdf

75	95	32	96	97	25	5	20	47	95
34	10	99	39	39	21	20	49	26	58
48	93	15	55	97	80	14	26	70	85
86	84	68	8	78	17	86	85	55	74
82	52	88	90	4	46	46	46	11	92
56	42	27	70	31	29	26	50	25	7
86	83	39	21	30	61	76	87	90	50
67	17	1	23	50	93	38	42	8	32
9	42	23	8	67	62	53	94	15	90
4	55	4	49	99	99	55	68	25	73

Histograms



h=30



h=10

Histograms

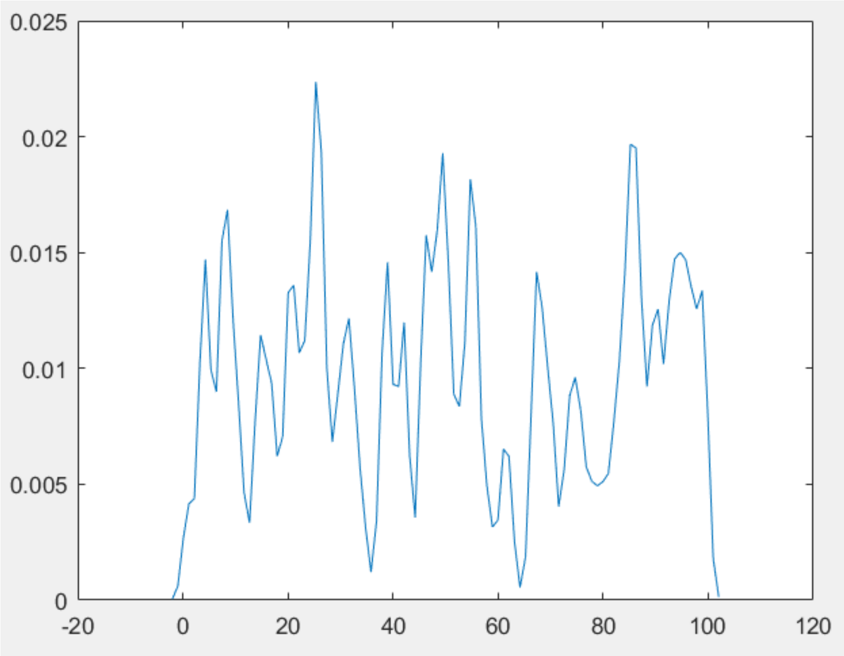
- It discretizes the density function
- The parameter *h* controls the amount of smoothing

- The number of bins grows exponentially as the dimensionality of the data increases
- The density function has discontinuities at the bin bounaries

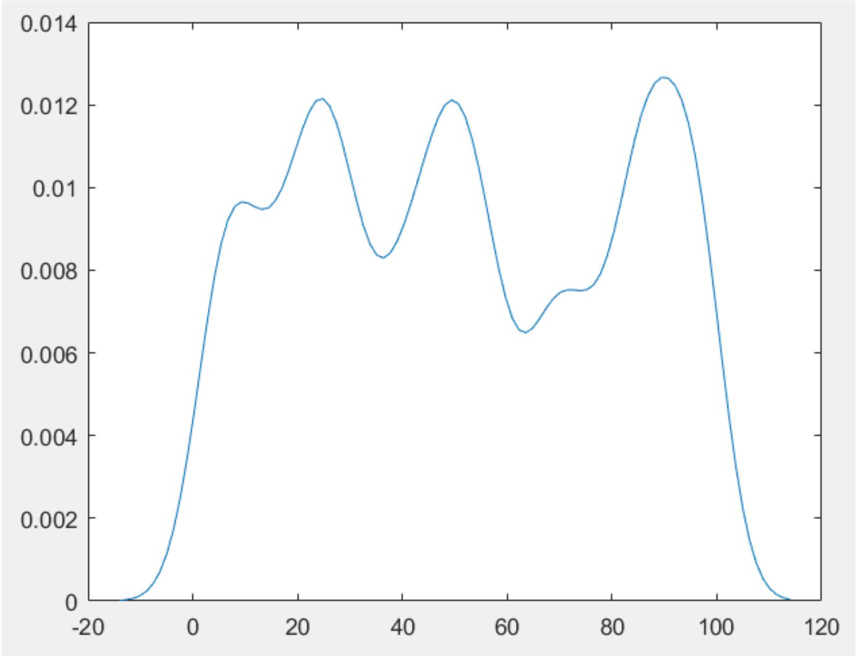
Kernel Estimator

- $\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)$
  - h: window width (smoothing parameter, bandwidth)
  - *K is a kernel function*
    - $\int_{-\infty}^{\infty} K(x)dx = 1$
  - Intuitively,
    - Place a bump at each observation
    - Kernel estimator is a sum of bumps

Kernel Estimator



h=1



h=5

Kernel Estimator

- $\hat{f}(x)$ inherits all the continuity and differentiability properties of K
- e.g., when K is the normal density function, $\hat{f}(x)$ will be a smooth curve
- The parameter h controls the amount of smoothing

Nearest Neighbor Method

- Adapt the amount of smoothing to local density of data

- Generalized kth nearest neighbor density estimate
- $\hat{f}(t) = \frac{1}{nd_k(t)} \sum_{i=1}^{n} K\left(\frac{t-X_i}{d_k(t)}\right)$
- $d_k(t)$: kth nearest distance from the query point, t

Variable Kernel Method

- $\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{hd_{i,k}}K\left(\frac{x-X_i}{hd_{i,k}}\right)$
- $d_{i,k}$: distance from X_i to the kth nearest point
  - Note that the window width is independent from query points

- The window width of the kernel placed on the point X_i is proportional to $d_{i,k}$
- It considers local density of observed data

General Weight Function

- $\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n} w(X_i, t)$
  - $\int_{-\infty}^{\infty} w(x, y) dy = 1$
  - $w(x, y) \geq 0$ for all x and y

  - e.g. histogram
    - $w(x, y) = \frac{1}{h(x)}$ if x and y are in the same bin
    - $w(x, y) = 0$ otherwise

Discussion

- Choice of Kernels
  - Epanechnikov
    - $K(t) = \frac{0.75\left(1-\frac{1}{5t^2}\right)}{\sqrt{5}}$ for $|t| < \sqrt{5}$
    - $K(t) = 0$ otherwise
  - Biweight
    - $K(t) = \frac{15}{16}(1 - t^2)^2$ for $|t| < 1$
    - $K(t) = 0$ otherwise
  - Gaussian
    - $K(t) = \frac{1}{\sqrt{2\pi}} e^{-0.5t^2}$

Discussion

- Choice of Smoothing Parameters
  - Manual choices: visualize density estimation results with different parameters and select a proper one
  - Automatic choices: select optimal parameters that minimize errors

Error Estimation for Density Estimation

- Mean Squared Error (MSE) at a single point, *t*
  - $MSE_t(\hat{f}) = E\left(\hat{f}(t) - f(t)\right)^2$
  - $MSE_t(\hat{f}) = \left(E\left(\hat{f}(t)\right) - f(t)\right)^2 + V\left(\hat{f}(t)\right)$
  - $E\left(\hat{f}(t)\right) = \frac{1}{n}\sum E\left(w(X_i, t)\right) = \int w(x,t) f(x) dx$ (Silverman 1986, 36pp)
  - $V\left(\hat{f}(t)\right) = \frac{1}{n} V\left(w(X_i, t)\right)$
    - $= \frac{1}{n}\left[\int w(x,t)^2 f(x) dx - \{\int w(x,t) f(x) dx\}^2\right]$
  - The bias, $E\left(\hat{f}(t)\right) - f(t)$, does not depend on the sample size, *n*
  - The variance, $V\left(\hat{f}(t)\right)$, decreases as we use more samples

Error Estimation for Density Estimation

- Asymptotic version (Silverman 1986, 39pp) for univariate data
  - $bias_h(x) = E\left(\hat{f}(x)\right) - f(x) \approx \frac{1}{2}h^2 f''(x)k_2$ *(derived using Taylor expansion)*
  - $V\left(\hat{f}(x)\right) \approx n^{-1}h^{-1}f(x)\int K(t)^2\,dt$

- Trade-off between bias and variance

- In practice,
  - We need a way to estimate unknown terms $f(x)$, $f''(x)$

Discussion

- Radiance estimation in (progressive) photon mapping is an application of the density estimation

- The radius of the radiance estimation is related to the error terms
- Q. Can we choose an optimal radius in a data-driven way?